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A Basis for the Analysis of Solid Continua Using the Integrated Force Method

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Introduction

ANALYSIS of the solid continuum involves solution of the field equations along with prescribed boundary conditions. Methods of analysis can be classified into three broad categories. In the first method, displacements are considered as the primary unknowns. Compatibility is satisfied automatically by imposing a continuity condition on displacements and their derivatives. The field equations are equilibrium equations (EE) in terms of displacements. All boundary conditions can be readily expressed in displacements. This approach is usually called the stiffness method (SM). In the stiffness formulation, both the field equations and all the boundary conditions are expressed in terms of displacements. In the second approach, stresses are treated as primary variables. Equilibrium in the field and on the boundary are satisfied by imposing Cauchy's differential constraints in stresses. Field equations are essentially St. Venant's compatibility conditions. In this formulation, a complete solution in terms of stresses is obtained, provided all boundary conditions are prescribed in terms of stresses. This approach is called the Flexibility Method (FM). This approach is not so convenient, particularly when some of the boundary conditions are in terms of displacements because boundary displacements cannot be easily converted to primary stress variables. The third approach is the mixed formulation and is intended primarily to overcome the difficulty in the flexibility method. Reissner-Hellinger and Washizu-Yu principles provide the basis for many of the mixed formulations currently available in the literature.

Thus, a fundamental question arises: "Is it possible to develop a formulation for mixed boundary-value problems such that the solution to stresses can be completely obtained without any recourse to displacements in the field or on the boundary?" The mixed boundary-value problem can be solved completely in terms of displacements, because the boundary stress conditions can be readily expressed in terms of displacements. Unfortunately, when stresses are considered as primary variables, the expressions for boundary displacements in terms of these primary variables (stresses) can not be readily determined.

In this context, a recent formulation called Integrated Force Method (IFM), originally developed for discrete systems,[1-6] is examined. A variational basis for this method was also given in Refs. 7 and 8. The novelty of this approach is the establishment of "boundary compatibility conditions" (BCC) analogous to the stress boundary conditions of Cauchy. In a series of publications concerned with the analysis of discrete structures,[1-6] it has been demonstrated that the complete solution can be obtained in terms of stresses, using the boundary compatibility conditions without any reference to the displacements. In this Note we examine the continuum analogue of the IFM to explore and identify the class of mixed boundary-value problems in stressed solid continuum, which can be handled entirely in terms of stresses.

Basic Theory of the Mixed Boundary-Value Problem

A typical two-dimensional mixed boundary-value problem of elasticity is shown in Fig. 1. On the boundary segment S_σ forces are prescribed, whereas displacements are constrained on the remaining portion of the boundary S_u. The variational principle of the integrated force method for the mixed boundary-value problem given in Ref. 2 can be restated as

$$\delta\Pi_s = \delta\Pi_s^{(1)} + \delta\Pi_s^{(2)} \tag{1}$$

where

$$\delta\Pi_s^{(1)} = \iint(\sigma_x\delta\epsilon_x + \sigma_y\delta\epsilon_y + \sigma_{xy}\delta\epsilon_{xy})\,\mathrm{d}x\,\mathrm{d}y + \delta\int_{S_u}\{\lambda_1(u-\bar{u}) + \lambda_2(v-\bar{v})\}\,\mathrm{d}\gamma \tag{2}$$

$$\delta\Pi_s^{(2)} = \iint(\epsilon_x\delta\phi_{,yy} + \epsilon_y\delta\phi_{,xx} - \epsilon_{xy}\delta\phi_{,xy})\,\mathrm{d}x\,\mathrm{d}y + \delta\int\{\mu_1(P-\bar{P}_x) + \mu_2(P-\bar{P}_y)\}\,\mathrm{d}\gamma \tag{3}$$

where u, v are the displacements, ϕ is the Airy's stress function, and x . . . etc. indicate differentiation with respect to the variable indicated. The body forces and initial strains are omitted for simplicity. The first part forms the basis of the traditional displacement formulation, with prescribed displacements on S_u and zero forces on S_σ. In this discussion we will consider only the second part $\Pi_s^{(2)}$. The stationary condition of the functional $\Pi_s^{(2)}$ yields the following equations for a rectangular domain:

Field Equation:

$$\nabla^4\phi = 0 \tag{4}$$

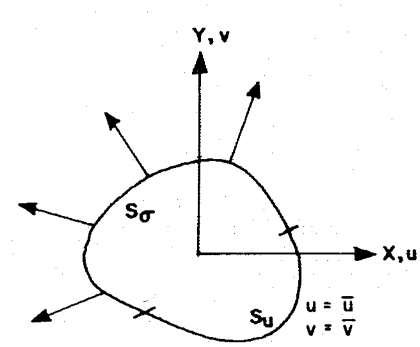

Fig. 1 A typical plane stress problem.

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a) Boundary Conditions (along the edges, x = const):

$$\sigma_x = \phi_{,yy} = \bar{P}_x$$
$$\tau_{xy} = -\phi_{,xy} = \bar{P}_y \tag{5}$$

b) Boundary Compatibility Conditions:

$$\epsilon_y = 1/E\,(\phi_{,xx} - \nu\phi_{,yy}) = 0$$
$$\epsilon_{y,x} - \gamma_{xy,y} = \{\phi_{,xx} - 2(1+\nu)\,\phi_{,yy}\}_{,x} = 0 \tag{6}$$

Equation (6) may also be rewritten in terms of displacements as

$$\epsilon_y = v_{,y} = 0$$
$$\epsilon_{y,x} - \gamma_{xy,y} = u_{,yy} = 0 \tag{7}$$

which represent a constant value of v and a linear variation of u (constant rotation). Since consideration $u = v = 0$ on this boundary satisfies Eq. (7), the stress field obtained as a solution of Eq. (4) with boundary conditions Eqs. (5) and (6) corresponds to the solution of the problem with homogeneous boundary conditions on S_u. The boundary conditions on the edge y = constant have similar form.

Thus, the stresses in this class of mixed boundary-value problems in which displacement boundary conditions are homogeneous can be completely formulated in terms of stresses as solution to Eqs. (4–6). This process eliminates the consideration of displacement boundary conditions on S_u.

Example

As an example we consider the bending of a beam fixed at both ends ($x = 0$ and a). It is subjected to a uniformly distributed load of intensity q. The potential function for this example can be expressed as follows:

$$\delta\Pi_s = \delta\Pi_s^{(1)} + \delta\Pi_s^{(2)}$$

where

$$\delta\Pi_s^{(1)} = \int_o^a M\delta k\,\mathrm{d}x - \int_o^a q\,\delta\omega\mathrm{d}x + \delta(\lambda,\omega)\Big|_o^a + \delta(\lambda_2\omega_{,x})\Big|_o^a \tag{8}$$

$$\delta\Pi_s^{(2)} = \int_o^a k\,\delta M\,\mathrm{d}x + \delta\int_o^a \lambda(M_{,xx} - q)\,\mathrm{d}x \tag{9}$$

where M is the bending moment and k is the curvature. In what follows we consider $\Pi_s^{(2)}$. M and k may be expressed in terms of a stress function ϕ as

$$M = \phi_{,xx}$$
$$k = -\phi_{,xx}/EI \tag{10}$$

where E is Young's Modulus and I is the second moment of the area. The stress function ϕ, required to satisfy the equilibrium equation

$$M_{,xx} = q \quad \text{or} \quad \phi_{,xxxx} = q \tag{11}$$

is introduced in the potential function through the Lagrangian multiplier λ.

Using Eqs. (10) and (11), the expression for $\delta\Pi_s^{(2)}$ is rewritten as

$$\delta\Pi_s^{(2)} = -\int_o^a (\phi_{,xx}/EI)\,\delta\phi_{,xx}\,\mathrm{d}x + \delta\int_o^a \lambda(\phi_{,xxxx} - q)\,\mathrm{d}x$$

Using the standard variational procedure, the governing differential equations and boundary conditions are obtained as

$$\phi_{,xxxx} = q \tag{12}$$

$$[(\phi/EI) + \lambda]_{,xxxx} = 0 \tag{13}$$

with boundary conditions

$$\phi = 0;\ \phi_{,x} = 0 \ \text{at}\ x = 0 \ \text{and}\ a \tag{14}$$

$$\lambda = 0;\ \lambda_{,x} = 0 \ \text{at}\ x = 0 \ \text{and}\ a \tag{15}$$

Equations (12–15) represent the IFM. It may be readily verified that the solution of Eq. (12) along with the boundary conditions of Eq. (14) completely defines ϕ and hence the moment distribution in this beam. Thus, the moment response of the clamped beam with displacement boundary conditions can be obtained using the integrated force method without any reference to displacements on the boundary or in the field. Furthermore, the solution of Eq. (13) along with the boundary conditions of Eq. (15) yields the Lagrangian multiplier, which can be readily identified as the normal displacement distribution in the beam.

The present study is an attempt to establish the feasibility of obtaining complete information on stresses in a mixed boundary-value problem with homogeneous displacement boundary conditions. In principle, this can be extended to include body forces and elastic supports, etc., by appropriately including them in either part of the potential function given in Eq. (1). In such a case, the decoupling of the two problems cannot always be expected. Further study in this direction will be interesting and useful.

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On a General Method of Vibration Analysis in Curvilinear Coordinates

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MOST of the dynamic problems in elasticity in curvilinear coordinates are formulated to solve a particular type of

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